1 Burger's Model

The Burger's model illustrated in Figure 1.1 simulates creep mechanisms in *PFC^{2D}*. The model contains the Kelvin model and the Maxwell model, which are connected in series in both the normal and shear directions, respectively, at a contact point.

The model has the following properties.

bur_knk normal stiffness for Kelvin section (K_{k_n})

bur_cnk normal viscosity for Kelvin section (C_{k_n})

bur_knm normal stiffness for Maxwell section (K_{m_n})

bur_cnm normal viscosity for Maxwell section (C_{m_n})

bur_ksk shear stiffness for Kelvin section (K_{k_s})

bur_csk shear viscosity for Kelvin section (C_{k_s})

bur_ksm shear stiffness for Maxwell section (K_{m_s})

bur_csm shear viscosity for Maxwell section (C_{m_s})

bur_fric friction coefficient (f_s)

bur_notension switch (0: with tensile force (default); 1: without tensile force)

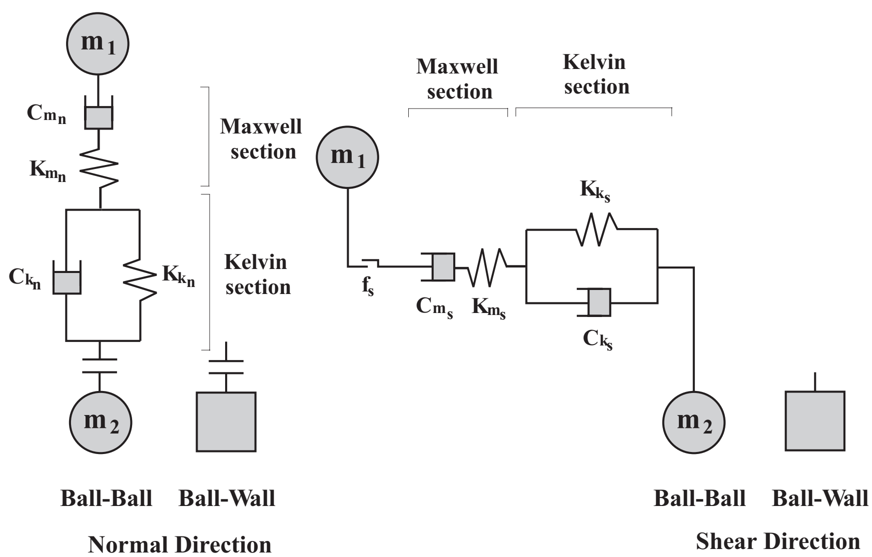

Figure 1.1 The Burger's model in PFC

1.1 Numerical Scheme

The total displacement of the Burger's model, u, is the sum of the displacement of the Kelvin section (u_k) and that of the Maxwell section (u_{m_K},u_{m_C}) of the model, given by Eq. (1.1). Note that symbols $\pm$ and $\mp$ correspond to the cases of normal direction and shear direction, respectively. (For example, $\pm$ means + for normal direction and - for shear direction.)

$$u = u_k + u_{m_K} + u_{m_C} \tag{1.1}$$

The first and second derivatives of Eq. (1.1) are given by Eqs. (1.2) and (1.3):

$$\dot{u} = \dot{u}_k + \dot{u}_{m_K} + \dot{u}_{m_C} \tag{1.2}$$

$$\ddot{u} = \ddot{u}_k + \ddot{u}_{m_K} + \ddot{u}_{m_C} \tag{1.3}$$

The contact forces, f, using the Kelvin section and the first derivative, are given by Eqs. (1.4) and (1.5):

$$f = \pm K_k u_k \pm C_k \dot{u}_k \tag{1.4}$$

$$\dot{f} = \pm K_k \dot{u}_k \pm C_k \ddot{u}_k \tag{1.5}$$

Also, using stiffness K_m and viscosity C_m of the Maxwell section,

$$f = \pm K_m u_{m_K} \tag{1.6}$$

$$\dot{f} = \pm K_m \dot{u}_{m_K} \tag{1.7}$$

$$\ddot{f} = \pm K_m \ddot{u}_{m_K} \tag{1.8}$$

$$f = \pm C_m \dot{u}_{m_C} \tag{1.9}$$

$$\dot{f} = \pm C_m \ddot{u}_{m_C} \tag{1.10}$$

Using Eqs. (1.2) through (1.10), the second-order differential equation for contact force f is given by

$$f + \left[\frac{C_k}{K_k} + C_m \left(\frac{1}{K_k} + \frac{1}{K_m} \right) \right] \dot{f} + \frac{C_k C_m}{K_k K_m} \ddot{f} = \pm C_m \dot{u} \pm \frac{C_k C_m}{K_k} \ddot{u} \tag{1.11}$$

From Eq. (1.4) of the Kelvin section,

$$\dot{u}_k = \frac{-K_k u_k \pm f}{C_k} \tag{1.12}$$

By using a central difference approximation of the finite difference scheme for the time derivative and taking average values for u_k and f,

$$\frac{u_k^{t+1} - u_k^t}{\Delta t} = \frac{1}{C_k} \left[-\frac{K_k (u_k^{t+1} + u_k^t)}{2} \pm \frac{f^{t+1} + f^t}{2} \right] \tag{1.13}$$

therefore,

$$u_k^{t+1} = \frac{1}{A} \left[B u_k^t \pm \frac{\Delta t}{2C_k} \left(f^{t+1} + f^t \right) \right] \tag{1.14}$$

where

$$A = 1 + \frac{K_k \Delta t}{2C_k} \tag{1.15}$$

$$B = 1 - \frac{K_k \Delta t}{2C_k} \tag{1.16}$$

For the Maxwell section, the displacement and the first derivative are given by Eqs. (1.17) and (1.18):

$$u_m = u_{m_K} + u_{m_C} \tag{1.17}$$

$$\dot{u}_m = \dot{u}_{m_K} + \dot{u}_{m_C} \tag{1.18}$$

Substituting Eqs. (1.7) and (1.9) into Eq. (1.18),

$$\dot{u}_m = \pm \frac{\dot{f}}{K_m} \pm \frac{f}{C_m} \tag{1.19}$$

By using a central difference approximation of the finite difference scheme and taking the average value for f,

$$\frac{u_m^{t+1} - u_m^t}{\Delta t} = \pm \frac{f^{t+1} - f^t}{K_m \Delta t} \pm \frac{f^{t+1} + f^t}{2C_m} \tag{1.20}$$

therefore,

$$u_m^{t+1} = \pm \frac{f^{t+1} - f^t}{K_m} \pm \frac{\Delta t (f^{t+1} + f^t)}{2C_m} + u_m^t \tag{1.21}$$

The total displacement and the first derivative of the Burger's model are given by Eqs. (1.22) and (1.23):

$$u = u_k + u_m \tag{1.22}$$

$$\dot{u} = \dot{u}_k + \dot{u}_m \tag{1.23}$$

By using the finite difference scheme for the time derivative,

$$u^{t+1} - u^t = u_k^{t+1} - u_k^t + u_m^{t+1} - u_m^t \tag{1.24}$$

Substituting Eqs. (1.14) and (1.21) into Eq. (1.24), the contact force, f^{t+1}, is given by Eq. (1.25):

$$f^{t+1} = \pm \frac{1}{C} \left[u^{t+1} - u^t + \left(1 - \frac{B}{A}\right) u_k^t \mp D f^t \right] \tag{1.25}$$

where

$$C = \frac{\Delta t}{2C_k A} + \frac{1}{K_m} + \frac{\Delta t}{2C_m} \tag{1.26}$$

$$D = \frac{\Delta t}{2C_k A} - \frac{1}{K_m} + \frac{\Delta t}{2C_m} \tag{1.27}$$

Contact force f^{t+1} can be calculated from known values for u^{t+1}, u^t, u_k^t and f^t.

1.2 Simple Examples

Figure 1.2 shows the time history of the contact force in the normal direction for step input $u(t) = 0.01$. The *FISH* program is listed in Example 1.1, in which two balls are created under an overlap condition, then fixed in position and spin. The result shows that the contact force exponentially decreases as time increases, which coincides with the analytical solution, Eq. (1.28), as shown by Figure 1.3.

$$f(t) = A_1 \exp(z_1 t) + A_2 \exp(z_2 t) \tag{1.28}$$

where

$$A_1 = \frac{b_2 z_1 + b_1}{a_2(z_1 - z_2)} \tag{1.29}$$

$$A_2 = \frac{b_2 z_2 + b_1}{a_2(z_2 - z_1)} \tag{1.30}$$

$$a_1 = \frac{C_k}{K_k} + C_m\left(\frac{1}{K_k} + \frac{1}{K_m}\right) \tag{1.31}$$

$$a_2 = \frac{C_k C_m}{K_k K_m} \tag{1.32}$$

$$b_1 = \pm C_m \tag{1.33}$$

$$b_2 = \pm \frac{C_k C_m}{K_k} \tag{1.34}$$

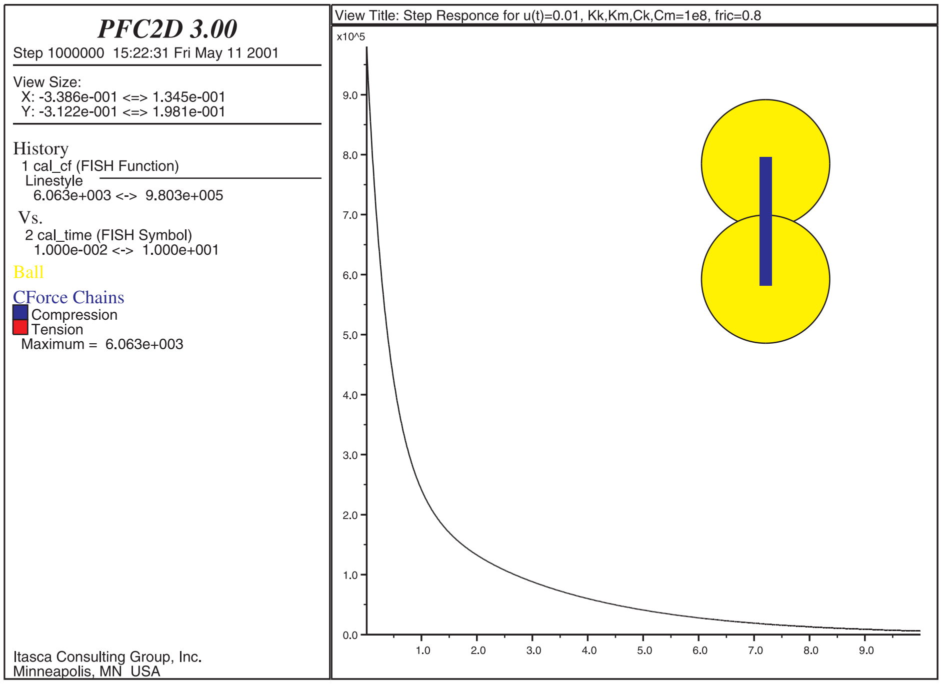

Figure 1.2 Time history of normal contact force with two balls, PFC^{2D}

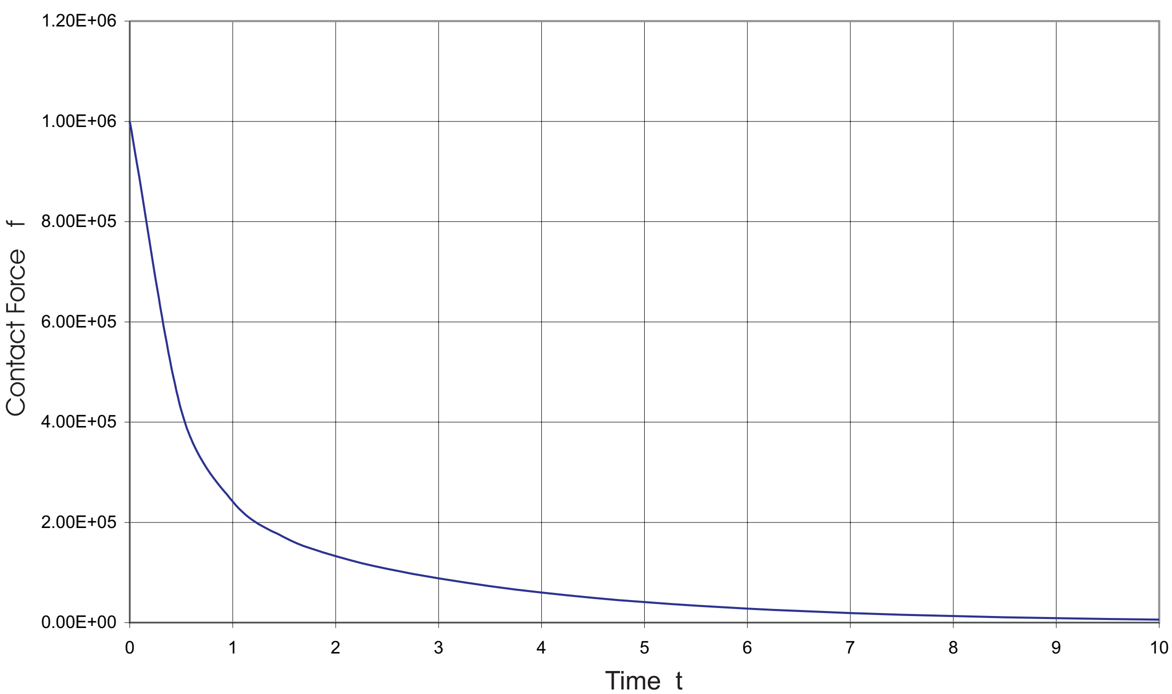

Figure 1.3 Analytical solution for step input $u(t) = 0.01$, ($K_k = K_m = C_k = C_m = 1.0 \times 10^8$)

Example 1.1 Data file for simple stress relaxation test

```
; fname: burger2d.DAT
new
set dt max 1e-5
;----- FISH function -------
def set_param
  k_set = 1.0e8
  c_set = 1.0e8
  f_set = 0.0
end
set_param
;--------------------------
def catch_contact_bur
  cp = fc_arg(0);
  c_model(cp) = ’burger’
  c_prop(cp,’bur_knk’)  = k_set
  c_prop(cp,’bur_cnk’)  = c_set
  c_prop(cp,’bur_knm’)  = k_set
  c_prop(cp,’bur_cnm’)  = c_set
  c_prop(cp,’bur_ksk’)  = k_set
  c_prop(cp,’bur_csk’)  = c_set
  c_prop(cp,’bur_ksm’)  = k_set
  c_prop(cp,’bur_csm’)  = c_set
  c_prop(cp,’bur_fric’) = f_set
end
;--------------------------
def cal_cf
  cal_cf = c_nforce(contact_head)
  cal_time = time - time0
end
;--------------------------
def reset_time
  time0 = time
end
; ------- main ---------------
; config & load DLL for the Burger’s model
config cppudm
model load bur2wrv ; 2d VC++ Release version
; model load bur2wdv ; 2d VC++ Debug version
; set the Burger’s model
model burger
set fishcall 6 catch_contact_bur
set fishcall 0 cal_cf
; make BALL
ball id = 1 rad = 0.05 x = 0.0 y = 0.0
ball id = 2 rad = 0.05 x = 0.0 y = 0.09
```

```
prop dens = 2600
fix x y spin
; set property
prop bur_knk = k_set bur_cnk = c_set
prop bur_knm = k_set bur_cnm = c_set
prop bur_ksk = k_set bur_csk = c_set
prop bur_ksm = k_set bur_csm = c_set
prop bur_fric = f_set
; history data
his id=1 cal_cf
his id=2 cal_time
hist nstep = 1000
; cycle until 10 s
reset_time
cycle 1000000
```